Report IRB-StP-GR-260596

An Example of Exact Solution in the LTB Model

by

Alexander Gromov

St. Petersburg State Technical University

Faculty of Technical Cybernetics, Dept. of Computer Science

29, Polytechnicheskaya str. St.-Petersburg, 195251, Russia

and

Istituto per la Ricerca di Base

Castello Principe Pignatelli del Comune di Monteroduni

I-86075 Monteroduni(IS), Molise, Italia

e-mail: gromov@natus.stud.pu.ru

Abstract

The Caushy problem in the LTB model is formulated. The rules of calculating three undetermined functions which defined a solution in the LTB model are presented. One example of exact nonhomogeneous model is studied. The limit transformation to the FRW model is shown.

PACS number(s):98.80

1 The Introduction

The LTB model is one of the most known spherical symmetry model in general relativity. It was created by Lemaitre [1], Tolman [2], and Bondy [3] during the period of time from 1933 to 1947. The exact solution have been obtained by Bonnor [4] in 1972 and [5] in 1974. The LTB model represented one of the simplest nonhomogeneous nonstationary cosmological models and due to this fact is used to study some new ideas in the cosmology.

The present interest to LTB was risen by the observations shown the fractal structure of the Universe in the large scale [6], [7]. The review of the topics in Cosmology is presented in [8]. In a set of papers the LTB model is used to study the observational datas and redshift as a main cosmological test [9] - [14].

The central problem of using the LTB model is in calculation of three undetermined functions which defined the solution. There is a set of ways how to solve this problem from the physical point of view in the mentioned papers. This article is devoted to the mathematical point of view on this matter.

2 The LTB Model

This section is devoted to presentation the Lemaitre-Tolman-Bondy model and section 2 of the paper [2] is cited. The co-moving system of coordinate is used in this model where the interval has the form

$$ds^2(r,t) = -e^{\lambda(r,t)}dr^2 - e^{\omega(r,t)}\left(d\theta^2 + sin^2\theta d\phi\right) + dt^2. \tag{1}$$

$\lambda(r,t)$ and $\omega(r,t)$ are metrical functions [2] definding the solution. In the co-moving system of coordinate with the line element (1) the energy-momentum tensor

$$T^{\alpha,\beta} = \rho\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{2}$$

has only one non zero component

$$T^4_4 = \rho \qquad T^\alpha_\beta = 0, \qquad \alpha \quad \text{or} \quad \beta = 4.$$

Using them together with Dingle results [15] we obtain the system of equations of the LTB model:

$$8\pi T^1_1 = e^{-\omega} - e^{-\lambda}\frac{\omega'^2}{4} + \ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \Lambda = 0 \tag{3}$$

$$8\pi T^2_2 = 8\pi T^3_3 =$$

$$-e^{-\lambda}\left(\frac{\omega''}{2} + \frac{\omega'^2}{4} - \frac{\lambda'\omega'}{}\right) + \frac{\ddot{\lambda}}{4} + \frac{\dot{\lambda}^2}{4} + \frac{\ddot{\omega}}{2} + \frac{\dot{\omega}^2}{4} + \frac{\dot{\lambda}\dot{\omega}}{4} - \Lambda = 0 \tag{4}$$

$$8\pi T^4_4 = e^{-\omega} - e^{-\lambda}\left(\ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \frac{\lambda'\omega'}{2}\right) + \frac{\dot{\omega}^2}{2} + \frac{\dot{\lambda}\dot{\omega}}{2} - \Lambda = 8\pi\rho \tag{5}$$

$$8\pi e^\lambda T^1_4 = -8\pi T^4_1 = \frac{\omega'\dot{\omega}}{2} - \frac{\dot{\lambda}\omega'}{2} + \dot{\omega}' = 0, \tag{6}$$

where

$$' = \frac{\partial}{\partial r} \qquad \dot{} = \frac{\partial}{\partial t} \tag{7}$$

The equation (6) has the solution

$$e^\lambda = e^\omega\frac{\omega'^2}{4f^2(r)}, \tag{8}$$

where $f(r)$ is undetermined function. Substituting (8) into (3) we obtain

$$e^{\omega}\left(\ddot{\omega}+\frac{3}{4}\dot{\omega}^2-\Lambda\right)+\left[1-f^2(r)\right]=0. \tag{9}$$

This equation is integrated twice. First integral gives the equation

$$e^{3\,\omega/2}\left(\frac{\dot{\omega}^2}{2}-\frac{2}{3}\Lambda\right)+2e^{\omega/2}\left[1-f^2(r)\right]=F(r), \tag{10}$$

and the second one gives the equation

$$\int\frac{\mathrm{d}e^{\omega/2}}{\sqrt{\mathrm{f}^2(\mathrm{r})-1+\frac{1}{2}F(\mathrm{r})\mathrm{e}^{-\omega/2}+\frac{\Lambda}{3}\mathrm{e}^{\omega}}}=t+\mathbf{F}(\mathrm{r}) \tag{11}$$

The equations (10) and (11) hold undetermined functions $F(r)$ $\mathbf{F}(r)$. The substitution of (8) into (5) together with (10) gives the equation for density

$$8\pi\rho=\frac{1}{\omega' e^{3\omega/2}}\frac{\partial F(\mathrm{r})}{\partial \mathrm{r}} \tag{12}$$

3 The Cauchy Problem for the LTB Model

Before we study the LTB model let us introduce the follow characteristic values: a velocity of light c, an observational meaning of the Habble constant $\tilde{H}$, a characteristic time $1/\tilde{H}$ and characteristic length $c/\tilde{H}$. We use the co-moving system of coordinates in the LTB model, so the radial coordinate r has a sense of Lagrangian mass coordinate [16],[17]. Two dimensionless variables μ and τ are definded by the rules

$$\mu=\frac{r}{m} \qquad \tau=\tilde{H}t, \tag{13}$$

where m is a full mass of the "gas". The dimensionless Habble function $h(\mu,\tau)$ and density $\delta(\mu,\tau)$ will be also used:

$$h(\mu,\tau)=\frac{H(\mu,\tau)}{\tilde{H}}, \qquad \delta(\mu,\tau)=\frac{\rho(r,t)}{\rho_0}, \tag{14}$$

where $\rho_0=\rho(0,0)$, $H(0,0)=\tilde{H}$.

Let us write the interval (1) as

$$ds^2(r,t)=-Ae^{\lambda(r,t)}dr^2-Be^{\omega(r,t)}\left(d\theta^2+sin^2\theta d\phi\right)+c^2dt^2, \tag{15}$$

where two constants A and B are introduced to take into account the fact that (15) is dimension equation.

The dimension of $\left[ds^2\right]$ is L^2, dimension of $[A]$ is L^2M^{-2} and dimension of $[B]$ is L^2, so

$$A=\left(\frac{c}{\tilde{H}m}\right)^2, \qquad B=\left(\frac{c}{\tilde{H}}\right)^2. \tag{16}$$

The interval (15) has now the form

$$\left(\frac{\tilde{H}}{c}\right)^2 ds^2(r,t)=-e^{\lambda(r,t)}dr^2-e^{\omega(r,t)}\left(d\theta^2+sin^2\theta d\phi\right)+d\tau^2, \tag{17}$$

Together with metrical functions $\omega(\mu,\tau)$ and $\lambda(\mu,\tau)$, introduced by Tolman, it is conveniently to use the Bonnor's function [4]

$$R(\mu,\tau)=e^{\omega(\mu,\tau)/2}. \tag{18}$$

In the Bonnor's notation the interval (17) takes the form

$$\left(\frac{\tilde{H}}{c}\right)^2 ds^2(\mu,\tau) = -\frac{\left[R'(\mu,\tau)\right]^2}{f^2}d\mu^2 - R^2(\mu,\tau)\left(d\theta^2 + sin^2\theta d\phi\right) + d\tau^2 \tag{19}$$

As it is shown in [16] and [17], the Bonnor's coordinate $R(\mu,\tau)$ has a sense of Euler coordinate, so the equation (18) correlates geometrical radius of the sphere $R(\mu,\tau)$ where the particle is located, and the Lagrangian coordinate μ of this sphere.

To describe the radial motion we will use the Habble function connected with variation of the length dl:

$$h = \frac{d\dot{l}}{dl}, \tag{20}$$

where, according to the (17), for dl^2 we read:

$$dl^2 = e^{\lambda(\mu,\tau)}d\mu^2 = \left(\frac{R'}{f}d\mu\right)^2. \tag{21}$$

By the substitution (21) into the definition of the Habble function (22) we obtain

$$h(\mu,\tau) = \frac{\dot{\lambda}(\mu,\tau)}{2} = \frac{\dot{R}'}{R'} = \frac{\partial \ln R'}{\partial\tau} \tag{22}$$

By the integration of the equation (22) we obtain the formula for metrical function $\lambda(\mu,\tau)$:

$$\lambda(\mu,\tau) = 2\int\limits_0^\tau h(\mu,\tau)d\tau + \lambda(\mu,0) \tag{23}$$

A solution in the LTB model is defined by the functions $f(r)$, $F(r)$ and $\mathbf{F}(r)$. These functions are obtained in the process of solution of the system of PDE, so to define them the initial/boundary conditions should definitely be used. The metrical function $\omega(\mu,\tau)$ is the solution of the equation (11). The equations of the LTB model are obtained in [2] and solved in the parametric form for the three cases $f^2(\mu) < 1$, $f^2(\mu) = 1$ and $f^2(\mu) > 1$ in [4] and [5].

The equations (9) - (11) are valid for every τ, and due to this fact in the Cauchy problem they *definde* the functions $f^2(\mu)$, $F(\mu)$ and $\mathbf{F}(\mu)$ at the moment of time $\tau = 0$:

$$\mathrm{f}^2(\mu) - 1 = \mathrm{e}^{\omega_0(\mu)}\left(\ddot{\omega}_0(\mu) + \frac{3}{4}\dot{\omega}_0^2(\mu) - \Lambda\right) \tag{24}$$

$$F(\mu) = \mathrm{e}^{3\,\omega_0(\mu)/2}\left(\frac{\dot{\omega}_0^2(\mu)}{2} - \frac{2}{3}\Lambda\right) + 2\mathrm{e}^{\omega_0(\mu)/2}\left[1 - \mathrm{f}^2(\mu)\right] \tag{25}$$

$$\mathbf{F}(\mu) = \int\limits_{\mathrm{e}^{\omega_0(0)/2}}^{\mathrm{e}^{\omega_0(\mu)/2}} \frac{de^{\tilde{\omega}_0(\mu)/2}}{\sqrt{f^2(\mu) - 1 + \frac{1}{2}F(\mu)e^{-\tilde{\omega}_0(\mu)/2} + \frac{\Lambda}{3}e^{\tilde{\omega}_0(\mu)}}} \tag{26}$$

At the time $\tau = 0$ the equation (8) *defindes* the function $\lambda_0(\mu)$:

$$e^{\lambda_0(\mu)} = e^{\omega_0(\mu)}\frac{[\omega_0(\mu)]'^2}{4f^2(\mu)}. \tag{27}$$

Substituting (24) into (25), we obtain

$$F(\mu) = \mathrm{e}^{3\omega_0(\mu)/2}\left(-2\ddot{\omega}_0(\mu) - \dot{\omega}_0^2(\mu) + \frac{4}{3}\Lambda\right). \tag{28}$$

Comparing (8) - (10) with (24) - (25), we find out that

$$f^2(\mu)-1=e^{\omega_0(\mu)}\left(\ddot{\omega}_0(\mu)+\frac{3}{4}\dot{\omega}_0^2(\mu)-\Lambda\right) \quad = \quad e^{\omega(\mu,\tau)}\left(\ddot{\omega}(\mu,\tau)+\frac{3}{4}\dot{\omega}^2(\mu,\tau)-\Lambda\right) \tag{29}$$

and

$$F(\mu)=e^{3\omega_0(\mu)/2}\left(-2\ddot{\omega}_0(\mu)-\dot{\omega}_0^2(\mu)+\frac{4}{3}\Lambda\right) \quad = \quad e^{3\omega(\mu,\tau)/2}\left(-2\ddot{\omega}(\mu,\tau)-\dot{\omega}^2(\mu,\tau)+\frac{4}{3}\Lambda\right) \tag{30}$$

are not dependent on time. Let's use the previous results to calculate the functions $\mathbf{F}(\mu)$ and integral in the equation (11). Substituting the definitions (24) and (28) into (26) we obtain:

$$\mathbf{F}(\mu)=\pm\int\limits_{\omega_0(0)}^{\omega_0(\mu)}\frac{d\tilde{\omega}}{\dot{\tilde{\omega}}}. \tag{31}$$

The function $\mathbf{F}(\mu)$ is equal to zero at the moment of time $\tau=0$ according the definition. Substituting the right part of the equations (29) and (30) into the (26), we obtain the equation

$$\pm\int\limits_{\omega(\mu,0)}^{\omega(\mu,\tau)}\frac{d\tilde{\omega}}{\dot{\tilde{\omega}}}=\pm\int\limits_{\omega_0(0)}^{\omega_0(\mu)}\frac{d\tilde{\omega}}{\dot{\tilde{\omega}}}+\tau. \tag{32}$$

This analysis of the LTB model shows that the functions

$$\left.\begin{array}{c}\omega(\mu,\tau)|_{\tau=0}=\omega_0(\mu) \quad \dot{\omega}(\mu,\tau)|_{\tau=0}=\dot{\omega}_0(\mu)\\ \ddot{\omega}(\mu,\tau)|_{\tau=0}=\ddot{\omega}_0(\mu),\end{array}\right\} \tag{33}$$

and constants

$$\left.\begin{array}{c}\omega(\mu,0)|_{\mu=0}=\omega_0(0) \quad \dot{\omega}(\mu,0)|_{\mu=0}=\dot{\omega}_0(0)\\ \ddot{\omega}(\mu,0)|_{\mu=0}=\ddot{\omega}_0(0), \quad \Lambda,\end{array}\right\} \tag{34}$$

are included into the definitions (24) - (26) and they form the initial conditions of the Cauchy problem for the equations (3) - (6). In accordance with (27) the function $\lambda_0(\mu)$ is not include in the set of initial conditions.

Substituting (28) into the (12), we obtain the general expression for the density of "gas" in the LTB model:

$$8\pi\rho(\mu,\tau)=\frac{\mathrm{e}^{\frac{3}{2}[\omega_0(\mu)-\omega(\mu,\tau)]}}{\omega'(\mu,\tau)}\times \left\{3\left[\omega_0(\mu)\right]'\left[-\ddot{\omega}_0(\mu)-\frac{1}{2}\dot{\omega}_0^2(\mu)+\frac{\Lambda}{6}\right]-2\left[\ddot{\omega}_0(\mu)\right]'-2\dot{\omega}_0(\mu)\left[\dot{\omega}_0(\mu)\right]'\right\} \tag{35}$$

The function $\omega(\mu,\tau)$ from the equation (35) is the solution of the equation [2].

4 The $f^2=1$, $\Lambda=0$ solution

The solution of the main equation of the LTB model (11) will be obtained in this section for one special case

$$f^2=1, \qquad \Lambda=0. \tag{36}$$

This case allows simple analytical solution. The law of the density, Habble function, and cosmological parameter will be also studied here.

The condition $f^2(\mu) = 1$, as it goes from the (34), correlates the initial conditions as follows:

$$\ddot{\omega}_0(\mu) + \frac{3}{4}\dot{\omega}_0^2(\mu) = 0. \tag{37}$$

Due to this reason, the general number of the initial condition (34) is decreased by one unit. To obtain the solution we'll start with the solution of the equation (10). ¿From it it's obvious that:

$$\left(\frac{\partial e^{\omega/2}}{\partial \tau}\right)^2 = f^2 - 1 + \frac{1}{2}Fe^{-\omega/2} + \frac{\Lambda}{3}e^{\omega}. \tag{38}$$

In the general case the function F is defined by the equation (25). Whit the help of (36) it takes the form

$$F = \frac{\dot{\omega}_0^2}{2}e^{3\omega_0/2}. \tag{39}$$

The equation (38) with conditions (36) takes the form

$$e^{3\omega_0/4}\frac{\partial e^{3\omega/4}}{\partial \tau} = \pm\frac{3}{4}\dot{\omega}_0. \tag{40}$$

Integrating (40) we obtain

$$e^{\frac{3}{4}[\omega-\omega_0]} = \pm\frac{3}{4}\dot{\omega}_0\tau + \mathbf{F}(\mu). \tag{41}$$

We find out the function $\mathbf{F}$ from the initial conditions then time $\tau = 0$:

$$\mathbf{F} = 1. \tag{42}$$

The solution now is:

$$e^{\frac{3}{4}[\omega-\omega_0]} = 1 \pm \frac{3}{4}\dot{\omega}_0\tau. \tag{43}$$

In the Bonnor's notation the equation (44) becomes as follows:

$$\frac{R(\mu,\tau)}{R_0(\mu)} = \left(1 \pm \frac{3}{4}\dot{\omega}_0(\mu)\tau\right)^{2/3}, \tag{44}$$

where the upper sign correspond to the expansion into the infinity from the initial condition and lower sign correspond to the collapse from one. The solution with "-" describes time of collapse depening on the initial mass coordinate: if the particle has a Euler coordinate $R_0(\mu)$ at the moment $\tau = 0$, the time of collapse is equal to

$$\bar{\tau}(\mu, 0) = \frac{4}{3\dot{\omega}_0(\mu)} \tag{45}$$

We obtain the density corresponding to this solution by the substitution (43) into (35). Let's denote

$$\nu = \frac{[\dot{\omega}_0]'}{\omega_0{}'}. \tag{46}$$

$$8\pi\delta(\mu,\tau) = \frac{\dot{\omega}_0}{\left(1 \pm \frac{3}{4}\dot{\omega}_0\tau\right)^2} \cdot \frac{\nu + \frac{3}{4}\dot{\omega}_0}{1 \pm \dfrac{\nu\tau}{1 \pm \frac{3}{4}\dot{\omega}_0\tau}}, \tag{47}$$

Using the definition (22) we find out the Habble's function for the solution (43):

$$h(\mu,\tau) = \pm\frac{\frac{3}{4}\dot{\omega}_0 + \nu}{1 \pm \left(\frac{3}{4}\dot{\omega}_0 + \nu\right)\tau} \mp \frac{1}{4} \cdot \frac{\dot{\omega}_0}{1 \pm \frac{3}{4}\dot{\omega}_0\tau} \tag{48}$$

Knowing the law of the density and Habble's function we obtain now the formulum for the cosmological parameter. By the definition

$$\Omega = \frac{\rho}{\rho_c}, \quad \text{where} \quad \rho_c = \frac{3H_0^2}{8\pi G}, \tag{49}$$

where H_0 and ρ mean the Habble function and the density at the moment of the observation. Let's assume that this moment is $\tau = 0$. The density, the critical density and Habble function at this moment are:

$$8\pi\delta(\mu, 0) = \dot{\omega}_0 \left(\frac{3}{4}\dot{\omega}_0 + \nu\right), \tag{50}$$

$$\delta_c(\mu, 0) = \frac{3\tilde{H}^2}{8\pi G\rho_0} h^2(\mu, 0). \tag{51}$$

$$h(\mu, 0) = \pm\left(\frac{1}{2}\dot{\omega}_0 + \nu\right) \tag{52}$$

$$\Omega(\mu, 0) = \frac{G\rho_0}{3\tilde{H}^2}\dot{\omega}_0 \frac{\frac{3}{4}\dot{\omega}_0 + \nu}{\left(\frac{\dot{\omega}_0}{2} + \nu\right)^2} \tag{53}$$

These functions dependent on time as follow:

$$\delta_c(\mu, \tau) = \frac{3\tilde{H}^2}{8\pi G\rho_0} h^2(\mu, \tau). \tag{54}$$

$$\Omega = \frac{16G\rho_0}{3\tilde{H}^2}\dot{\omega}_0 \left(\frac{3}{4}\dot{\omega}_0 + \nu\right) \cdot \tag{55}$$

$$\frac{\left(1 \pm \frac{3}{4}\dot{\omega}_0\tau\right)\left[1 \pm \left(\frac{3}{4}\dot{\omega}_0 + \nu\right)\tau\right]}{\left\{\pm 4\left(1 \pm \frac{3}{4}\dot{\omega}_0\tau\right)\left(\frac{3}{4}\dot{\omega}_0 + \nu\right) \mp \dot{\omega}_0\left[1 \pm \left(\frac{3}{4}\dot{\omega}_0 + \nu\right)\tau\right]\right\}^2} \tag{56}$$

5 FRW model

Every nonhomogeneous solution of the LTB model must include the FRW model as a limited case when the density and Habble function are not dependent on space coordinate for all moments of time. Let's study in which case the present solution is reduced to the FRW model? Only one condition satisfied this request:

$$\omega_0 = const \qquad \text{so,} \quad \nu_0(\mu) = 0. \tag{57}$$

It goes from (46) and (57) that in this case

$$8\pi\delta = \frac{3}{4}\left(\frac{\dot{\omega}_0}{1 \pm \frac{3}{4}\dot{\omega}_0\tau}\right)^2, \tag{58}$$

$$h = \pm\frac{1}{2}\frac{\dot{\omega}_0}{1 \pm \frac{3}{4}\dot{\omega}_0\tau}. \tag{59}$$

¿From these equations it goes:

$$8\pi\delta = 3h^2, \tag{60}$$

We obtain the meaning of the ω_0 from (58):

$$\dot{\omega}_0 = \pm 2\sqrt{\frac{8\pi}{3}}. \tag{61}$$

$\nu = 0$ together with (61) give

$$\Omega = \frac{G\rho_0}{\tilde{H}^2}. \tag{62}$$

6 Acknowledgements

I'm grateful to Prof. Arthur D. Chernin for encouragement and discussion. Dr. Yurij Barishev has initiated my interest to the modern Cosmology as fractal structure of the Universe and interpretation of the observations by calculation redshift in the LTB model. This paper was financially supported by "COSMION" Ltd., Moscow.

References

[1] Lemaître, Ann. de la Soc.Scient. de Bruxelles **A53**, 51 (1933).

[2] R.C.Tolman, Proc.Nat.Acad.Sci (Wash), **20**,169

[3] H.Bondy, MNRAS **107**, p.p. 410 - 425 (1947).

[4] W.B.Bonnor, MNRAS (1972) **159**, 261 - 268

[5] W.B.Bonnor, MNRAS (1974) **167**, 55 (1974).

[6] P.H.Coleman and L.Pietronero, Physics Reports, **213**, 6, (1992).

[7] L.Pietronero, Physica, **144A**, 257 (1987).

[8] Yu.V.Baryshev, F.Sylos Labini, M.Montuori, and L.Pietronero, Vistas in Astronomy, **38**, 4 (1994).

[9] M.B.Ribeiro, ApJ.**388**,1 (1992).

[10] M.B.Ribeiro, ApJ.**395**,29 (1992).

[11] M.B.Ribeiro, ApJ.**415** (1993).

[12] J.W.Moffat and D.S.Tatarski, Phys.Rev.D **45**,10 (1992)

[13] J.W.Moffat and D.S.Tatarski, Proceeding of the XXIXth RENCONTRE DE MORIOND, Series:Moriond Workshops, Villars sur Ollon, Switzerland,January 22-29, 1994. Editions Frontieres.

[14] J.W.Moffat and D.S.Tatarski, preprint UTPT-94-19 (1994).

[15] Dingle, Proc.Nat.Acad. **19**, 559, 1933.

[16] Landau, Lifshits, "The Field Theory", Moscow, "Nauka", 1973.

[17] Gurevich and Chernin, "The Introduction into Cosmology", Moscow,"Mir", 1978.

[18] H.B. Dwight, "Tables of Integrals", NY, 1961.

[19] G.A.Corn, T.M.Corn, "Mathematical Handbook", McGraw-Hill Company, NY, 1968.